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Difference Equations in Statistical Mechanics. II. Solid-on-Solid Models in Two Dimensions

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A review and some new results are presented for the solid-on-solid models of wetting in two dimensions (i.e., line interfaces) with an emphasis on the difference equations arising in the transfer matrix calculations for these models. Methods for solving the appropriate difference equations exactly or approximately are surveyed, including specific results for short-range, long-range power-law, and applied field-like (binding) external potentials.

KEY WORDS: Wetting transitions; generating functions; continued fractions.

1. INTRODUCTION

Solid-on-solid (SOS) models have been extensively used to describe the properties of interfaces separating coexisting thermodynamic phases. For reviews consult, e.g., refs. 1 and 2. In two dimensions, numerous SOS model studies of the wetting transition and associated surface and interfacial phenomena have been reported in recent years.[1–20] The purpose of this review is to survey recent developments in the studies of the *difference equations* associated with the $2d$ SOS models, as well as report some new results. In an accompanying article,[21] we review difference equations arising in the studies of the $2d$ cluster statistics models.

In Section 2, we introduce the standard restricted SOS model[6,17] and describe its solution in the case of no external potential. The physical picture of the wetting transition via interface depinning is detailed and the critical-point aspects of the transition are reviewed. Various external potentials of interest in wetting studies are described in Section 3. With external

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potentials, the SOS models are no longer exactly solvable in general. Different methods of approaching the problem are introduced in Sections 3–5. Specifically, in Section 3 we outline the generating function method (see ref. 21 for a more detailed review). Differential equation approximation is considered in Section 4. Finally, Section 5 describes the continued fraction techniques.

Only a limited number of SOS models with external potentials can be solved in closed form. An example of a short-range potential is presented in Section 6. Section 7 is devoted to the $1/r$ long-range potential case.[18] Some open problems are summarized in Section 8.

2. RESTRICTED SOLID-ON-SOLID MODEL

In this section we introduce the *restricted* SOS model.[6,17] Consider a planar square lattice of Ising spins, ± 1, at positions (x, y) with integer $0 \leqslant x < \infty$, $|y| < \infty$, i.e., a semi-infinite half-plane geometry. The boundary spins are fixed at -1 for $x = 0$ and $+1$ for $x = \infty$, thus forcing an interface in the system. One can argue generally[22] that there will exist a *long contour* separating the region of predominantly $(-)$ magnetization near the wall at $x = 0$ from the region of $(+)$ magnetization for large x. The shape of the long contour can, in principle, be quite complicated. It is, however, generally accepted[1,22] that for studying wetting phenomena it is sufficient to account for the SOS subset of Ising configurations with a single long contour, with no overhangs or bubbles. Thus, we consider spin configurations with $n_y \geqslant 1$ leftmost spins (at $x = 0, 1, \ldots, n_y - 1$) in each fixed-$y$ row taking values -1, while the remaining spins (at $x = n_y, n_y + 1, \ldots$) taking values $+1$. (Here $y = 0, \pm 1, \pm 2, \ldots$ labels the lattice rows.) The interfacial energy is modeled by the Hamiltonian

$$H/kT = \sum_y [U\,|n_y - n_{y-1}| - W\delta_{1n_y} + E(n_y)] \tag{2.1}$$

Here $U > 0$ represents the surface tension contribution. Contact interactions attracting the interface to the wall at $x = 0$ are represented by the $W > 0$ term. The external potential is denoted by $E(n)$.

In (2.1), the difference $|n_y - n_{y-1}|$ can take on any value. However, it is mathematically convenient and physically acceptable to further restrict the model to configurations with $|n_y - n_{y-1}| = 0$ or 1 for all y. Detailed studies indicate[23] that such a *restricted* SOS model is identical with the unrestricted model in all the qualitative features of the wetting behavior. Let us introduce the notation

$$0 < u \equiv e^{-U} < 1, \qquad w = e^{W - E(1)} > 1 \tag{2.2}$$

where we assume $W > E(1)$, to avoid unilluminating mathematical complications. [As explained in Section 3, the physically interesting potentials are usually considered in the regime $W \gg |E(1)|$.] Also, denote by n and m the n_y values in two consecutive rows. Then the transfer matrix T can be defined to have elements

$$T_{nm} = u^{|n-m|} w^{\delta_{1,n}} e^{-E(n)} (\delta_{0,n-m} + \delta_{1,|n-m|}) \tag{2.3}$$

Note that we chose a nonsymmetric transfer matrix. Let g_m denote the right-eigenvector elements. Then the eigenvalue equations

$$\sum_{m=1}^{\infty} T_{nm} g_m = \lambda g_n \tag{2.4}$$

reduce to

$$g_n + u(g_{n-1} + g_{n+1}) = \lambda g_n e^{E(n)}, \qquad n > 1 \tag{2.5}$$

which is a *second-order difference equation,* with the *boundary condition*

$$w(g_1 + u g_2) = \lambda g_1 \tag{2.6}$$

In order to illustrate the mechanism of the wetting transition, we will now solve (2.5)–(2.6) with no external potential, i.e., $E(n) \equiv 0$.[6] It is convenient to introduce two new variables t and ε defined by

$$\frac{1}{u} = \frac{1}{u_c} - \frac{w}{w-1} t \qquad \text{with} \quad u_c \equiv \frac{w-1}{2-w} \tag{2.7}$$

$$\lambda = 1 + 2u + 2u\varepsilon \tag{2.8}$$

The general solution of (2.5) with $E(n) \equiv 0$ is

$$g_n = A\gamma^n + B\gamma^{-n} \qquad \text{for} \quad \varepsilon \neq 0, -2 \tag{2.9}$$

$$g_n = A\gamma^n + Bn\gamma^n \qquad \text{for} \quad \varepsilon = 0, -2 \tag{2.10}$$

where

$$\gamma \equiv 1 + \varepsilon - [\varepsilon(2+\varepsilon)]^{1/2} \tag{2.11}$$

For $\varepsilon > 0$, we have $\gamma < 1$. On physical grounds we discard the exponentially growing term in (2.9), i.e., $B = 0$. The eigenvector $g_n \propto \gamma^n$ is then dominated by the "nonwet" spin configurations, with the layer of $(-)$ spins extending the distance

$$\xi_\perp = (-\ln \gamma)^{-1} \tag{2.12}$$

from the wall. However, the boundary condition (2.6) "quantizes" the nonwet part of the spectrum, yielding at most one eigenvalue. One can show (see refs. 6 and 17, and the next section) that this nonwet solution exists only for $u < u_c(w)$, corresponding to $t < 0$ in (2.7).

The eigenvectors corresponding to $-2 \leqslant \varepsilon \leqslant 0$ are dominated by the "wet" configurations with an unbounded $(-)$ layer. (The range $\varepsilon < -2$ is of no physical interest.) Excluding the end points $\varepsilon = 0, -2$, which require special consideration, and which we omit here except to quote that $B(\varepsilon = 0) = 0$, $g_n(\varepsilon = 0) \equiv A$, we note that γ and $\gamma^{-1} \equiv \gamma^*$ become complex (and conjugate) for $-2 < \varepsilon < 0$, with $|\gamma| = 1$. The boundary condition then determines the ratio A/B; the "wet" spectrum is not quantized. It exists for all t, covering the λ range $1 - 2u \leqslant \lambda \leqslant 1 + 2u$.

The interfacial free energy f and the longitudinal correlation length $\xi_{||}$ of the system are given, as usual in transfer matrix calculations, in terms of the largest and second eigenvalues λ_0 and λ_1, with the corresponding $\varepsilon_{0,1}$ values, by

$$f = -\ln \lambda_0, \qquad \xi_{||}^{-1} = \ln(\lambda_0/\lambda_1) \tag{2.13}$$

One should also use λ_0 in (2.12) to obtain a definition of the transverse correlation length. For $t < 0$ $[u < u_c(w)]$, which corresponds to the nonwet regime, explicit calculation yields[6,23]

$$\lambda_0 = \frac{w}{2}\left[1 - \left(1 + \frac{4u^2}{w-1}\right)^{1/2}\right] \tag{2.14}$$

while $\lambda_1 \equiv 1 + 2u$. For small negative t, we find, by expanding (2.14),

$$\varepsilon_0 \approx \tfrac{1}{2}t^2 \tag{2.15}$$

Relations (2.7)–(2.8) can be used to obtain the following rather general small-t and ε_0 expansion for f in (2.13), valid to $O(t^2)$ and $O(\varepsilon_0)$:

$$f = \ln\frac{2-w}{w} - \frac{2(w-1)}{2-w}t - \frac{2(w-1)}{(2-w)^2}t^2 - \frac{2(w-1)}{w}\varepsilon_0 + \cdots \tag{2.16}$$

Note that this expansion does not depend on the particular form of $\varepsilon_0(t; w)$, e.g., (2.15). The *singular part* of the interfacial free energy is thus proportional to $-\varepsilon_0$ and is given by

$$f_{\text{sing}} = -\frac{w-1}{w}t^2 \tag{2.17}$$

for small, negative t. (Note that the "regular part" of f has no physical significance unless one takes special care to relate the SOS parameters to

the original Ising model formulation.[22]) For the correlation lengths, we use (2.12)–(2.13) to get

$$\xi_{||} \approx \frac{w}{w-1} t^{-2}, \qquad \xi_{\perp} \approx |t|^{-1} \tag{2.18}$$

As $t \to 0^-$, there is a wetting critical point corresponding to the depinning of the interface from the wall. For *positive* t, $\lambda_0 \equiv 1 + 2u$, with constant g_n, and the spectrum is continuous. Thus, formally we obtain

$$f_{\text{sing}} = 0, \qquad \xi_{||} = \infty, \qquad \xi_{\perp} = \infty \tag{2.19}$$

in the wet regime.

3. EXTERNAL POTENTIALS. GENERATING FUNCTION TECHNIQUE

The nature of the wetting transition in the SOS model depends[12] on the form of the external potential $E(n)$. The following choices of $E(n)$ are of particular interest.

Exponential short-range potentials, behaving for large n according to

$$E(n) \approx ce^{-an}, \qquad a > 0 \quad (n \gg 1) \tag{3.1}$$

Such exponential potentials are generated in the process of renormalization[24,25] of the wetting models with short-range forces. Analytic results for this case will be reported in Section 6.

Power-law long-range potentials, behaving for large n according to

$$E(n) \approx cn^{-\phi}, \qquad \phi > 0 \quad (n \gg 1) \tag{3.2}$$

Such potentials are of practical importance in $3d$ wetting,[26] and have been extensively studied for the $2d$ case[12,14,15,18,19]; see Sections 4 and 7.

Applied field-like binding potentials,

$$E(n) = cn^{\psi}, \qquad c > 0, \quad \psi > 0 \tag{3.3}$$

Potentials of this form always suppress the wetting transition (bind the interface). However, one can study the $c \to 0^+$ scaling behavior.[2,12] The choice $\psi = 1$, corresponding to the applied magnetic field, is of special interest and has been considered by many authors[2,3,8,9,12] within the differential equation approximation. However, no exact results are known for the difference equation in this case.

In actual calculations, it is convenient to "deexponentiate" the potential. For the long-range potentials of the type (3.2), one argues that the

nature of the wetting transition depends mostly on the long-range tail, while the short range features of the potential represent a perturbation of the contact W interaction in (2.1), provided $|E(n)| \ll W$ for $n = O(1)$. Thus, one can choose a *power-law potential*

$$E(n) \equiv \ln(1 + cn^{-\phi}), \qquad \phi > 0, \quad c \text{ small} \tag{3.4}$$

which satisfies (3.2). This form has been employed in ref. 12: see Section 4. In Section 7 we discuss an alternative choice of a *power-law potential,*

$$E(n) \equiv \ln\left[1 + \frac{c}{n(n+1)\cdots(n+\phi-1)}\right] \tag{3.5}$$

for *integer* $\phi = 1, 2, \ldots$ (and small c).

A similar line of reasoning for the exponential potentials (3.1), is somewhat ambiguous, since they are short-range all along. However, for mathematical convenience, we will use in Section 6 the choice of the *exponential potential*

$$E(n) \equiv \ln(1 + ce^{-an}), \qquad a > 0 \tag{3.6}$$

which allows derivation of analytic results and a detailed analysis for small c.

"Deexponentiation" of the binding potentials (3.3), i.e., using *modified binding potentials*

$$E(n) = \ln(1 + cn^{\psi}), \qquad \psi > 0, \quad c > 0 \tag{3.7}$$

obviously changes the large-n asymptotic form. However, it has been argued[3,12] that the *small-c* scaling behavior is not affected. Note that for the "magnetic field" case $\psi = 1$, the difference equation (2.5) with (3.7) is the recurrence relation for the Bessel functions. Thus, exact results can probably be obtained in the $\psi = 1$ case. We are not aware of such a study, and it is outside the scope of our review to explore this possibility here.

The *generating function method* for solving difference equations is well known,[27] and is reviewed in an accompanying paper.[21] Here we emphasize features specific for the SOS model applications. The generating function is defined by

$$G(z) = \sum_{n=1}^{\infty} g_n z^{n-1} = g_1 + g_2 z + g_3 z^2 + \cdots \tag{3.8}$$

Equation (2.5) is then multiplied by z^n and summed over $n = 2, 3, \ldots$. In some cases, one ends up with a closed form equation for $G(z)$. Specifically,

for potentials of the type (3.4), (3.5), and (3.7), with *integer* ϕ and ψ one obtains differential equations for $G(z)$, some of which will be further discussed in Section 7.

For the potentials (3.6) and (3.3) with $\psi = 1$, functional equations for $G(z)$ of the type considered in ref. 21 are obtained. We derived formal expressions for the generating functions in both cases (not given here). However, we were not able to analyze the eigenvalue spectrum. [The potential (3.6) is treated in Section 6 by a different method.]

In order to illustrate the generating function approach, including the quantization of the "nonwet" eigenvalues imposed by the boundary condition (2.6), we turn again to the simple solvable case, $E(n) \equiv 0$. Here and below we will be mostly interested in the nonwet regime of finite f_{sing}, $\xi_{||}$, and $\xi_{\perp}$ [while (2.18) is typical for the wet phase]. The appropriate equation for $G(z)$ is *algebraic*,

$$[u(1+z^2)+(1-\lambda)z]\, G(z) = [u+(1-\lambda)z]\, g_1 + uzg_2 \tag{3.9}$$

By using (2.8) and (2.11), we can represent this as

$$G(z) = g_1 \frac{1 - z(2 + 2\varepsilon - g_2/g_1)}{(z-\gamma)(z-\gamma^{-1})} \tag{3.10}$$

The nonwet solution corresponds to $g_n \to 0$ for large n. Since g_n are the Taylor coefficients of $G(z)$, we conclude that two conditions must be satisfied. First, $\varepsilon > 0$ is needed to have real $\gamma < 1$. Second, the singularity at $z = \gamma$ yielding exponentially divergent g_n must be canceled. However, the ratio g_2/g_1 can be replaced by

$$\frac{g_2}{g_1} = \frac{w(1+t)+2\varepsilon}{w} \tag{3.11}$$

as implied by the boundary condition (2.6), with (2.7)–(2.8). Canceling the pole at γ therefore yields the relation between ε and t that determines the "quantized" eigenvalue $\varepsilon_0(t; w)$, see (2.14)–(2.15). Note that since the original difference equations (2.5) and (2.6) are linear, $G(z)$ has an arbitrary coefficient g_1 in (3.10).

4. DIFFERENTIAL EQUATION APPROXIMATION

Let us introduce the notation

$$E(n) \equiv \ln\left[1 + \frac{w-1}{w} V(n)\right] \tag{4.1}$$

which effectively defines $V(n)$ for $n = 1, 2, 3, \ldots$. After some algebra Eq. (2.5) [with (2.7), (2.8), (4.1)] can be expressed as

$$-(g_{n+1} - 2g_n + g_{n-1}) + \left[1 + \frac{2(w-1)}{w}\varepsilon - t\right] V(n)\, g_n$$
$$= (-2\varepsilon)\, g_n, \qquad n > 1 \tag{4.2}$$

A standard procedure for the critical region near the wetting transition (or rounded transition in the case of binding potentials), i.e., for *small* t and ε, is to approximate (4.2) by the differential equation

$$-\frac{\partial^2 g(X)}{\partial X^2} + V(X)\, g(X) = (-2\varepsilon)\, g(X) \tag{4.3}$$

where $0 \leqslant X < \infty$ is a continuous counterpart of n. Indeed, the fluctuations become large near the transition, and the magnetization profile varies over large distances (comparable to $\xi_\perp$). Thus, the discreteness of the original problem will be "washed out." [The small $O(t, \varepsilon)$ terms have been discarded in the coefficient of the potential.]

The boundary condition (3.11) is written as

$$\frac{g_2 - g_1}{g_1} = t + \frac{2}{w}\varepsilon \tag{4.4}$$

and is replaced by

$$\frac{g'(0)}{g(0)} = t \tag{4.5}$$

Neglecting the ε term is justified since for sharp, continuous wetting transitions $\varepsilon \sim |t|^{2-\alpha}$ with $\alpha < 1$. For first-order wetting transitions ($\alpha = 1$) and for binding potentials, more care may be required. In summary, (4.3) is a quantum mechanical Schrödinger equation with potential $V(X)$ and with the boundary condition that corresponds to a pointlike attracting (for $t < 0$) delta-function potential at the origin, $2t\delta(X)$. The $V(X) = 0$ wetting transition corresponds to the disappearance of the bound state as $t \to 0^-$. [To make this interpretation precise, one should restrict consideration to even wave functions and extend the problem to $-\infty < X < \infty$ symmetrically, i.e., with $V(|X|)$].

A more *ad hoc* approach[12] is to define the $2d$ SOS model by the quantum mechanical (QM) Hamiltonian with a potential consisting of a hard wall at $X = 0$, followed by a potential well at, say, $0 < X < b$, and

parameters adjusted to have one loosely bound state. Long-range potentials of the form (3.4), (3.6) are then introduced, i.e.,

$$V(X) = CX^{-\phi}, \qquad \phi > 0 \tag{4.6}$$

or

$$V(X) = CX^{\psi}, \qquad \psi > 0, \quad c > 0 \tag{4.7}$$

with $C \equiv cw/(w-1)$. Another approach[2] leading to somewhat different QM calculations consists of defining the SOS model with continuous interface height variables.

It should be emphasized that both the discrete and continuous SOS models in all their varieties supposedly approximate the original Ising problem to the extent of describing the wetting transition singularities. Thus, there is no *a priori* classification by the degree of approximation. We believe, however, that in the case of the first-order transitions[15,18,19] and for the description of interfacial pinning by the binding potentials deep in the wet regime ($t > 0$) the discrete models are more appropriate and the physical interpretation of their parameters is more transparent.

Although we do not intend to review in detail all the QM results available in the literature,[2,8,12–15,19] let us mention some conclusions of general interest. Consider first the power-law potentials (4.6). For $\phi > 2$, the mechanism of the wetting transition by the disappearance of the bound state into the continuum is not changed.[12] Nonanalytic corrections to scaling are present, e.g., in (2.15); however, the leading order dependence $f_{\text{sing}} \sim -t^2$ remains unchanged (with a modified t). For $0 < \phi < 2$ and $c < 0$, the wetting transition is suppressed: the potential is strong enough to pin the interface to the wall. In the spectral language, there are always bound states in addition to the continuous spectrum. For $t \gtrsim 0$, one finds[12] $f_{\text{sing}} \sim |c|^{2/(2-\phi)}$ for small $|c|$, up to possible logarithmic corrections[18] for $t \simeq 0$. Limited results for $c > 0$ have been outlined in ref. 15. Recently, a detailed study[18] of the $\phi = 1$ case has been reported for the discrete difference equation; see Section 7. It transpires that the wetting transition is first order for $c > 0$, with some unusual properties, e.g., divergent correlation lengths (at least for $\phi = 1$).

For the binding potentials (4.7), the spectrum is always discrete. Thes no wetting transition. For $t \gtrsim 0$, one has[12,14] $f_{\text{sing}} \sim c^{2/(2+\psi)}$. The important case of the applied magnetic field, $\psi = 1$, has also been considered in ref. 2. Their formulation does not involve the "deexponentiation" of the potential; see (4.1). Detailed results, including the c, $t \to 0$ crossover scaling forms for various quantities, are available in some cases.[2,8,12–15]

Finally, the borderline case $\phi = 2$ in (4.6) has been studied in detail recently.[19] A rich phase diagram with nonuniversal critical, multicritical, and first-order transitions has been discovered.

5. CONTINUED FRACTION METHOD

A linear second-order difference equation can be solved *formally*[28,29] in terms of continued fractions. Thus, we introduce the ratios

$$R_n \equiv g_{n+1}/g_n \qquad (n \geqslant 1) \tag{5.1}$$

so that R_1 is given by the right-hand side of (3.11). The difference equation (4.2) is then divided by g_n and after some rearrangement of terms is expressed as

$$R_{n-1} = \frac{1}{2(1+\varepsilon) + \{1 + [2(w-1)/w]\varepsilon - t\}\, V(n) - R_n}, \qquad n \geqslant 2 \tag{5.2}$$

This can be iterated to generate a continued fraction expansion for R_k ($k \geqslant 1$). Specifically, for R_1 we obtain [see (3.11)]

$$1 + t + \frac{2}{w}\varepsilon \tag{5.3}$$

$$= \cfrac{1}{2(1+\varepsilon) + \tilde{V}(2) - \cfrac{1}{2(1+\varepsilon) + \tilde{V}(3) - \cfrac{1}{2(1+\varepsilon) + \tilde{V}(4) - \cfrac{1}{2(1+\varepsilon) + \tilde{V}(5) - \cdots}}}}$$

where

$$\tilde{V}(n) \equiv \left[1 + \frac{2(w-1)}{w}\varepsilon - t\right] V(n) \tag{5.4}$$

Equation (5.3) is a formal implicit equation for $\varepsilon(t)$.

In connection with (5.3), we must consider the convergence of the continued fraction, implications of the general mathematical theory of continued fractions[30,31] for our problem, and the possibility of deriving specific results by utilizing (5.3). Generally, the second-order difference equation (4.2) has two linearly independent solutions, say $g_n^{(1)}$ and $g_n^{(2)}$. There is some arbitrariness in selecting the two solutions. However, if one can select them in such a way that

$$\lim_{n \to +\infty} [g_n^{(1)}/g_n^{(2)}] = 0 \tag{5.5}$$

then $g_n^{(1)}$ is termed the *minimal* solution.[28,29] The existence of the minimal solution is not granted. Indeed, our calculations in Section 2 for the case

$E(n) \equiv 0$ corresponding to $V(n) \equiv 0$ indicate that the minimal solution exists for $\varepsilon > 0$, in which case it is the physical "nonwet" solution $A\gamma^n$. However, none of the two solutions for $-2 < \varepsilon < 0$ is minimal. (On the borderline, $\varepsilon = 0$, the physical solution $g_n = A$ is minimal.) The differential equation approximation (Section 4) suggests that the *nonwet* solution being the minimal solution is a general rule. Indeed, it corresponds to the quantized localized ground-state eigenfunction in QM calculations, which decays at least exponentially as $X \to \infty$. On the other hand, the eigenfunctions of the continuous spectrum are linear combinations of two running waves, none of which is "minimal" as $X \to \infty$. (The properties of the borderline "zero-energy" solutions depend on the potential, specifically its long-range tail and sign.)

An important theorem by Pincherle[32] relates the convergence of the continued fraction for R_1 (and R_k with $k > 1$) to the existence of the minimal solution. Indeed, the right-hand side of (5.3), and similar continued fractions for $k > 1$, converge *if and only if* the difference equation possesses the minimal solution. Furthermore, the values of the continued fractions give R_k for the minimal solution, i.e., $g_n = g_n^{(1)}$ in (5.1). Thus, (5.3) is a well-defined equation for the ranges of the parameters t, w, and those of $V(n)$ for which the nonwet solution exists, and its free energy is given by the largest root $\varepsilon_0(t; w, \ldots)$.

The line of the argument can be reversed. The general mathematical theory[30,31] of the convergence of continued fractions can be invoked to make some of the conclusions on the spectrum of the problem more rigorous. For example, for the binding potentials (3.3), (3.7) corresponding to $V(n) \to \infty$ for large n, the appropriate types of continued fractions converge for all ε.[30,31] Thus, the boundary conditions will quantize all "energies" ε. However, for all other potentials introduced in Section 3, with $V(n) \to 0$, one can prove that the continued fraction converges for $\varepsilon > 0$ [and sometimes for $\varepsilon = 0$, depending on the details of $V(n)$], but it diverges for $-2 < \varepsilon < 0$. Thus, only the $\varepsilon > 0$ part of the physically relevant spectrum is quantized and can represent nonwet solutions.

The continued fraction in (5.3) is of the type called a J-fraction in the mathematical literature.[30,31] Unfortunately, not much is known about the analytic form of such fractions for $\varepsilon \to 0^+$. In the case of the potentials $V(n) \to 0$ (for large n), $\varepsilon = 0$ is a special point, since, as already mentioned, the J-fraction converges only for $\varepsilon > 0$ or $\varepsilon \geqslant 0$. This can be seen in the simplest case of no external potential. Indeed, for $V(n) \equiv 0$ the continued fraction is easily evaluated: for $\varepsilon \geqslant 0$, it converges to $\gamma(\varepsilon)$ [see (2.11)], and (5.3) reduces to

$$1 + t + \frac{2}{w}\varepsilon = 1 + \varepsilon - [\varepsilon(2 + \varepsilon)]^{1/2} \tag{5.6}$$

Thus, the continued fraction, and $t(\varepsilon)$, have an $\sim\varepsilon^{1/2}$ singularity as $\varepsilon\to 0^+$. Note that (5.6) yields (2.14). Specifically, for small ε, (5.6) is just $t\approx -(2\varepsilon)^{1/2}$, which has one nonwet solution (2.15), for $t<0$ only. Generally, the continued fraction equation (5.3) becomes an algebraic equation for $\varepsilon(t)$ if $V(n)$ is of finite range, i.e., $V(n>n_{\max})=0$.

6. EXPONENTIAL EXTERNAL POTENTIAL

In this section we consider the exponential potential

$$V(n)=Ce^{-an}, \qquad a>0 \tag{6.1}$$

In the notation of Eqs. (3.6), (3.1), and (4.1), $C=cw/(w-1)$. The difference equation (4.2) reads

$$g_{n+1}-2(1+\varepsilon)\,g_n+g_{n-1}=\left[1+\frac{2(w-1)}{w}\varepsilon-t\right]Ce^{-an}g_n \tag{6.2}$$

A discussion of difference equations of this type in ref. 21 (and references cited therein) leads to the following series for the *minimal* solution (we keep $\varepsilon>0$ here):

$$g_n=\gamma^n\sum_{m=0}^{\infty}p_m e^{-anm} \tag{6.3}$$

where the coefficients p_m satisfy the *first-order* recursion

$$\begin{aligned}&[\gamma(e^{-am}-1)+\gamma^{-1}(e^{am}-1)]\,p_m\\&\quad=C\left[1+\frac{2(w-1)}{w}\varepsilon-t\right]p_{m-1}, \qquad m>0\end{aligned} \tag{6.4}$$

obtained by substituting (6.3) in (6.2). With a convenient choice $p_0=1$, we get

$$\begin{aligned}p_m=C^m&\left[1+\frac{2(w-1)}{w}\varepsilon-t\right]^m\\&\times\prod_{k=1}^{m}[\gamma^{-1}(e^{ak}-1)-\gamma(1-e^{-ak})]^{-1}\end{aligned} \tag{6.5}$$

for $m=1, 2,\ldots$. The boundary condition [see (3.11)] reduces to

$$1+t+\frac{2}{w}\varepsilon=\gamma\,\frac{\sum_{m=0}^{\infty}p_m e^{-2am}}{\sum_{m=0}^{\infty}p_m e^{-am}} \tag{6.6}$$

This implicit equation for $\varepsilon(t)$ is rather complicated for general C. However, the series on the right-hand side are power series in C, since $p_m = O(C^m)$. Thus, for small C a systematic approximation scheme can be developed by accounting for corrections due to successively higher powers of C. The leading singularity ($\sim \varepsilon^{1/2}$) of $t(\varepsilon)$ still comes from the coefficient γ on the right-hand side of (6.6); see (2.11). Thus, the nature of the wetting transition is not changed (at least, for small C). For example, (2.15) is replaced by

$$\varepsilon_0 \approx (1/2 + \tau)(t - t_c)^2 \tag{6.7}$$

where the shifted t variable can be calculated to a desired accuracy for small C. For example, to $O(C)$,

$$t_c \approx -Ce^{-a}(e^a - 1)^{-1} \tag{6.8}$$

$$\tau \approx Ce^{-a}(e^a + 1)(e^a - 1)^{-2} \tag{6.9}$$

7. POWER-LAW EXTERNAL POTENTIALS. EXACT RESULTS FOR THE C/n POTENTIAL

In this section we discuss the long-range potentials defined by (3.5) for mathematical convenience. Thus, we have

$$V(n) = \frac{C}{n(n+1)\cdots(n+\phi-1)} \tag{7.1}$$

where $\phi = 1, 2, 3, \ldots$, and

$$C \equiv \frac{cw}{w-1} \tag{7.2}$$

Multiplication of the difference equation (4.2) by $Cz^{n-1}/V(n)$ and summing over $n = 2, 3, \ldots$ yields after some algebra the following differential equation for the generating function [see (3.8)]:

$$\frac{\partial^\phi}{\partial z^\phi}\left[z^{\phi-1}(z-\gamma)(z-\gamma^{-1})\,G(z)\right]$$

$$= C\left[1 + \frac{2(w-1)\varepsilon}{w} - t\right][G(z) - g_1] + (\phi!)[g_2 - (\gamma + \gamma^{-1})\,g_1] \tag{7.3}$$

where $\gamma(\varepsilon)$ is defined by (2.11). The solution of (7.3), with the additional condition (3.11), has been reported only for the case $\phi = 1$.[18] There is

some hope to make analytical progress for $\phi = 2$, since there is a mathematical literature on the appropriate homogeneous equation.[33]

The inhomogeneous term in (7.3) is a constant. Thus, additional z differentiation yields a homogeneous differential equation of order $\phi + 1$. Small-z analysis then indicates that out of its $\phi + 1$ linearly independent solutions, only *two* admit power-series expansion around $z = 0$. The conditions $G(0) = g_1$ and $G'(0) = g_2$ then determine the coefficients of the linear combination of these two solutions. Thus, we end up with

$$G(z) = g_1 f_1(z; \varepsilon, t, w, c) + g_2 f_2(z; \varepsilon, t, w, c) \tag{7.4}$$

For the wet regime, $-2 \leqslant \varepsilon \leqslant 0$, the singularities of $f_k(z)$ nearest to the origin will be on the unit circle, in fact, at $z = \gamma$ and $\gamma^{-1} \equiv \gamma^*$ (where $|\gamma| = 1$). The ratio g_2/g_1 is fixed by (4.4); however, there is no quantization of ε. In the nonwet regime, $\varepsilon > 0$, the additional condition of canceling, in $G(z)$, the singularity of $f_k(z)$ at $z = \gamma < 1$ to let the singularity at $z = \gamma^{-1}$ dominate the convergence of the power series (3.8) will lead to the quantization of ε.

General expectations for $\phi = 1, 2, 3, \ldots$ presented above have been checked[18] in detail by exact calculations for the simplest case $\phi = 1$. Consult ref. 18 for the explicit form of $f_k(z)$. The resulting equation for $\varepsilon_0(t; w, c)$ is rather complicated, and is not reproduced here. Scaling analysis for small t, ε, and c yields the following results. For $c < 0$ potentials, causing attraction of the interface to the substrate, the wetting transition is no longer sharp. The rounding is described asymptotically by the crossover scaling form

$$\varepsilon_0 \approx \bar{c}^2 P\left(\frac{t - t_c}{\bar{c}}\right) \tag{7.5}$$

where

$$\bar{c} = c w_0/(w_0 - 1) \qquad \text{with} \quad w_0 \equiv e^w \tag{7.6}$$

and

$$t_c = \frac{w_0}{w_0 - 1} c \ln |c| + \left[1 + \frac{w_0}{w_0 - 1}\left(\kappa + \ln \frac{w_0}{w_0 - 1}\right)\right] c + o(c) \tag{7.7}$$

with Euler's constant $\kappa = 0.5772156649\ldots$. Details on the form of the scaling function P and implications of (7.5) can be found in ref. 18. Note, in particular, the logarithmic nonscaling shift in t, which is an unusual feature.

For $c > 0$ potentials, which repel the interface, thus competing with the contact wall interaction, the wetting transition remains sharp. However, it becomes first-order, but with divergent correlation lengths. The scaling form (7.5) applies with a different scaling function P. However, $P \equiv 0$ for $t > t_c$, and P vanishes linearly as $t \to t_c^-$: one finds

$$\varepsilon_0 \approx \tfrac{3}{2}\bar{c}(t_c - t) \tag{7.8}$$

for small, fixed $c > 0$. This is reminiscent of a first-order transition, since the derivative $\partial\varepsilon_0/\partial t$ is discontinuous at t_c. However, for the correlation lengths $\xi_{||}$ and $\xi_\perp$ one finds a continuous divergence with new exponents $\nu_{||} = 1$ and $\nu_\perp = 1/2$ (different from $\nu_{||} = 2$, $\nu_\perp = 1$ for $c = 0$).

8. OPEN PROBLEMS. CONCLUDING REMARKS

Solid-on-solid models in two dimensions are a rich source of exact or nearly exact information on critical phenomena at wetting transitions. Results[1-20] already cover systems with second-order and first-order transitions, as well as rounded transitions (for binding potentials). In this section a brief list is given of some open problems, the resolution of which would be of particular benefit to the general theory of wetting.

Development of the mathematical theory of analytic properties of the continued fraction (5.3) for small ε is highly desirable. Indeed, in its present form the continued fraction method is useful for general considerations (see Section 5), but no tractable results of practical interest can be obtained.

For the power-law external potentials [see (3.2), (3.4), (3.5), (7.1)], a particularly interesting case is the borderline value $\phi = 2$ studied recently within the QM model.[19] It is also important to emphasize that the choice (7.1) for the potential is dictated by mathematical convenience. For C not too small, higher order power-law contributions to the difference $V(n) - Cn^{-\phi}$ may have a *qualitative* effect on the wetting transition[34,35] (if they are not small).

For the binding potentials (3.3), (3.7), the general scaling behavior at the rounded transition is known.[2,12] Here the most interesting case is that of the applied field, $\psi = 1$. Study of the $\psi = 1$ potential (3.3) beyond the QM-type approximations, i.e., for the difference equation and preferably without deexponentiating the potential, may yield interesting new insights. It is also interesting to investigate the applied field effect on wetting with long-range forces, e.g., on the first-order wetting transition induced by the C/r long-range potential (Section 7). Interesting *numerical* studies of applied field effects on solid-on-solid model correlation functions have been initiated by Stecki and Dudowicz (see ref. 36 and references cited therein).

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